AGT conjecture

Physical background

$\Gamma \subset SU(2)$ finite subgrp $\longleftrightarrow$ ADE Dynkin diagram

type IIB $/(\mathbb{C}^2/\Gamma) \rightsquigarrow$ 6d (2,0)-theory :

This theory gives a partion funct. Z_Γ

"path integral", but this theory has no lagrangian, ...

M : 6dim mfd (Riem. metric)

$\cup$

D : 4d submfd + something like gauge fields (bdry condition)

$\rightsquigarrow Z_\Gamma(M, D, \cdots) \in \mathbb{C}$ cpx number

Fix C : 2dim mfd + marked points $\{x_1, \cdots, x_n\}$ and

Take $\begin{cases} M = C \times X \\ D = \{x_1, \cdots, x_n\} \times X \end{cases}$ (X : 4 mfd)

Then we can consider $Z_\Gamma((C, \{x_i\}) \times ?)$: X^4 + gauge field $\longmapsto$ number

This gives a 4d quantum field theory.

We make a "Topological twist" in 6^d th. $\Rightarrow$ depends only on
– the conformal structure of $(C, \{x_i\})$
– topology of X + "function" on instanton moduli sp.

Hence we get a topological QFT in 4 dim.

Conversely we can fix 4 mfd X to get a 2^d CFT.
(We put 4^d gauge fields to punctures in 2^d)

<u>example</u>

Take C = torus with period τ

$\Longrightarrow$ $N=4$ SUSY gauge theory with group G_Γ

roughly $\sim$ $Z = \sum_n q^n \, e(\mathcal{M}(G_\Gamma, n))$ $\quad (q = e^{2\pi i \tau})$

$C_\tau \cong C_{-1/\tau} \Longrightarrow Z(\tau) = Z(-1/\tau) \times q^{\circledast}$

i.e., $Z(\tau)$ is a modular form

Vafa–Witten 1994

math • $X = \widetilde{\mathbb{C}^2/\Gamma}$ N 1992 $\bigoplus_n H_{mid}(\mathcal{M}(\mathcal{V}(k), n))$: int. h.w. rep. of $\hat{\mathfrak{g}}_\Gamma$

$\Gamma = A_{k-1}$ of level $= k$

$\Longrightarrow \sum q^n \text{"}e_{L^2}\text{"}(\mathcal{M}(\mathcal{V}(k), n)) =$ char. of

This has the modular inv. property.
(Kac–Peterson, 1984)

• X : cpx surface "$\mathcal{M}(\mathcal{V}(1), n)$" $=$ $\mathrm{Hilb}^n(X)$

$$\sum q^n e(\mathrm{Hilb}^n X) = \prod_{d=1} \frac{1}{(1-q^d)^{e(X)}}$$

Göttsche
(Dedekind η-fct.)

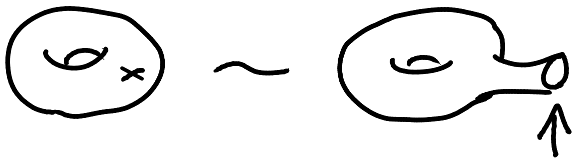
N, Grojnowski 1994 $\bigoplus_n H_*(\mathrm{Hilb}^n(X))$: rep. of Heisenberg alg $= \widehat{\mathfrak{gl}(1)}$

again char.

However, the proofs of these results do not give any clue why such results hold.

For example, I give generators as correspondences in moduli spaces, and check relations.

Better to look at $Z_\Gamma(X\times ?)$ as a 2^d CFT. (Segal's axioms)



- closed $C \longrightarrow$ number
- C with bdry $\longrightarrow$ vector in quantum Hilb sp. $Z(\underset{\text{"}S^1\text{"}}{\underline{\partial C}}) = \mathcal{H}$

In this framework partition fct for T^2 $=$ $\mathrm{char}(e^{2\pi i \tau}$ on $\mathcal{H})$

So my old result says

$$\left(\begin{array}{l}\Gamma = A_{k-1} \\ X = \widetilde{\mathbb{C}^2/\Gamma'}\end{array}\right. \Rightarrow \mathcal{H} = \text{a rep. of } \hat{\mathfrak{g}}_{\Gamma'} \text{ at level } k$$

Then the CFT, in question, must be WZW model for $\hat{\mathfrak{g}}_{\Gamma'}$ at level k

<u>Side remark</u> $(\text{type IIB}/\mathbb{C}^2/\Gamma)/\widetilde{\mathbb{C}^2/\Gamma'}$ This suggests a symmetry $\Gamma \longleftrightarrow \Gamma'$

For type A, this is level-rank duality

Braverman-Finkelberg : double affine Grassmannian $\Gamma' = A_{k-1} \rightsquigarrow \hat{\mathfrak{g}}_\Gamma$ of level k

AGT : $X = \mathbb{R}^4 \circlearrowleft T^2$ torus action (this is a substitute of Riem. metric)

Γ as before

But consider not $\bigoplus_n H_{mid}(\mathcal{M}(G_P, n))$, but

a larger space $\bigoplus_n H_*^{T^2 \times G_P}(\mathcal{M}(G_P, n))$.

$\Rightarrow$

The CFT is the one associated with $\mathcal{W}(\mathfrak{g}_P)$: W-algebra.

In particular, this leads to

<u>Conjecture</u> $\bigoplus_n H_*^{T^2 \times G_P}(\mathcal{M}(G_P, n))$ has a structure of a representation of $\mathcal{W}(\mathfrak{g}_P)$.

(highest wt rep. central charge $\leftarrow$ $H^*_{T^2}(pt) = \mathbb{C}[\varepsilon_1, \varepsilon_2]$

highest wt $\leftarrow$ $H^*_{G_P}(pt)$

Whittaker vector $\longleftrightarrow$ $\sum_n [\mathcal{M}(G_P, n)]$ etc type A : proved by Schiffmann–Vasserot, Maulik–Okounkov

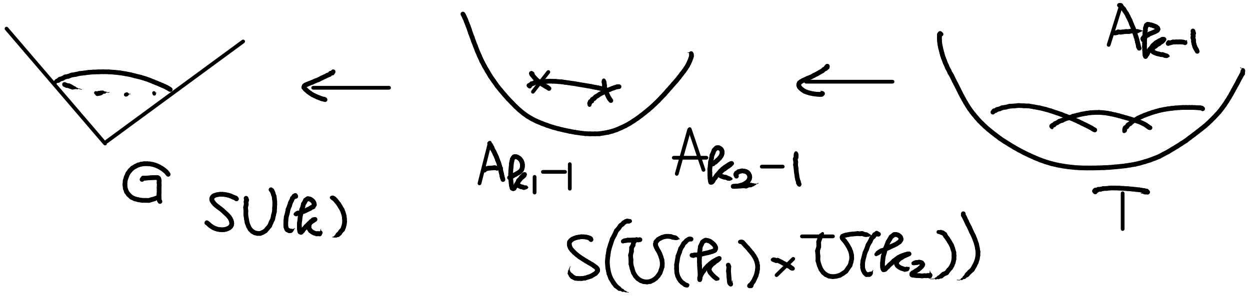
<u>Key ingredient</u>

Consider $G \supset L$ Levi subgroup $S(U(n_1)\times\cdots\times U(n_\ell))$

$$\begin{bmatrix} * & 0 & 0 \\ 0 & * & 0 \\ 0 & 0 & * \end{bmatrix}$$

If $G = G_\Gamma$, then this corresponds to a partial resol of $\mathbb{C}^2/\Gamma$

e.g.



type IIB / $\mathbb{C}^2/\Gamma$ type IIB / X type IIB / $\widetilde{\mathbb{C}^2/\Gamma}$

For each choice of $G \supset P \nearrow L$ (parabolic) $\mathcal{M}(G,n) \times \mathcal{M}(L,n) \cup \mathcal{L}(P,n)$ lagrangian subvar.

(In the algebro-geometric picture, this roughly corresponds to
G-b'dle $\hookleftarrow$ P-b'dle $\longrightarrow$ L-b'dle

$\sum_n [\mathcal{L}(P,n)]$ defines an operator $\bigoplus H_*^{T^2 \times G}(\mathcal{M}(G,n)) \to \bigoplus H_*^{T^2 \times L}(\mathcal{M}(L,n))$

More precisely, we need to assign multiplicities to irreducible components suitably.

$L = T \rightsquigarrow$ RHS $= \bigotimes$ Fock rep. of Heis.

So $\bigoplus H_*^{T^2 \times G}(\mathcal{M}(G,n))$ is a subspace in $\bigotimes$ Fock rep of Heis.

The corr. result in W-alg. is known Feigin-Frenkel

W-alg $\subset \bigotimes$ Fock rep

image $= \bigcap_i$ Ker(screening operator$_i$)